IMPLEMENTING HIERARCHICAL BAYESIAN MODEL TO FERTILITY DATA: THE CASE OF ETHIOPIA

Haftu Gebrehiwot[1], Stefano Mazzuco[2]

Second Bayesian Young Statisticians Meeting (BAYSM 2014)
Vienna, September 18–19, 2014

[1] University of Padua, Department of Statistical Sciences,Padova, Italy
<gebrehiwot@stat.unipd.it

[2] University of Padua, Department of Statistical Sciences,Padova, Italy
mazzuco@stat.unipd.it

Abstract

Modeling fertility curves has attracted the interest of demographers for many years. A variety of mathematical models have been proposed in demographic literature for modeling one-year the age specific fertility pattern of many population. In this study, we proposed to use the Skew Normal Distribution to fit the fertility schedules in Ethiopia, and showed that this proposed model is flexible enough for the patterns vis-á-vis other models using AIC as a model selection criterion. Bayesian hierarchical model through Gibbs sampler was then introduced under different prior specifications for parameters estimation and uncertainty analysis. Examples using simulated and application to the 2011 Ethiopian DHS fertility data are also considered for checking the validity of the proposed model.

Keywords: skew normal distribution; gibbs sampler; ASFR, fertility pattern, fertility rate.

1 Introduction

The age-specific fertility rate (ASFR hereafter) curve, in general, is a bell-shaped unimodal curve which first rises slowly and then sharply in the age group 15-19, attains its modal value somewhere between ages 20-29, declines first slowly and then steeply till it approaches zero around the age of 50 years even though some countries has already started showing a deviation from this classical bell shaped curve.A large number of models(namely: the Quadratic spline (QS), Cubic spline (CS), Beta function (BF), Gamma function (GF), Hadwiger function (HF), Skew Normal (SN), Gompertez curve (GC), Adjusred Error Model(AEM), Polynomial function (PF) and Model-1 of Peristera et.al., etc) have been proposed in demographical litratures for modeling the one-year age specific fertility curves of many populations, although fitting these models to curves of Ethiopian data has not been undertaken yet.To accurately model fertility patterns in Ethiopia, a mathematical model that is easily used, and provides good fit for the data is required.In this study, we proposed to use the Skew Normal Distribution to fit the fertility schedules, and in order to determine the performance of the proposed model, we conducted some preliminary analysis of fitting the model alongside with ten other commonly used models stated above.The criterion followed in fitting these models is the principle of nonlinear least squares.Results obtained from this preliminary analysis reveal that the values of the AIC for the proposed model, SN, is lowest for majority, i.e., in 6 of the 11 regions considered. This shows that this model is better able to reproduce the empirical fertility data of Ethiopia and its regions than the other existing models considered.

The article is organized as follows. In Section 2, we first provide a brief description of the skew-normal distributions and its properties. Section 3 is devoted to the derivation of the unconditional distribution of the parameters. We employ MCMC algorithm with a latent variable within the Gibbs Sampler to generate samples in Section 2.4. We perform some simulation studies to check the validity of our model in Section 3. In Section 4, we present the application of the proposed hierarchical structure to Fertility data of Ethiopia. Finally, we offer a brief discussion

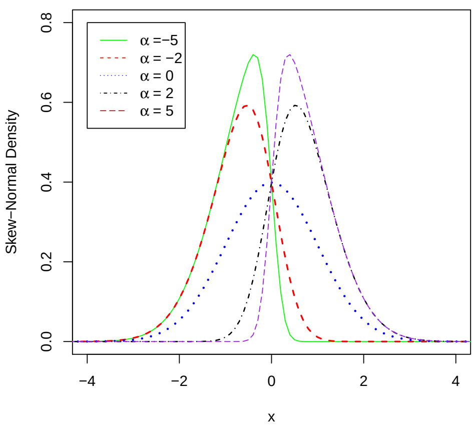
2 Bayesian Inference

2.1 Scalar Skew Normal Distribution

The skew-normal(SN) distribution, formally first introduced by Azzalini (1985), attracted a great deal of attention in the literature because of their flexibility in modeling skewed data, mathematical tractability and inclusion of the normal distribution as a special case. Thus, a random variable X is said to have a standard SN distribution, denoted by $X \sim \mathcal{SN}(\alpha)$ distribution if the probability density function (pdf) is of the form:

$$f(x; \lambda) = 2\phi(x)\,\Phi(\alpha x), \quad x \in \mathbb{R}, \quad \alpha \in \mathbb{R} \tag{1}$$

where ϕ and Φ denote the pdf and the cdf of standard normal distribution, respectively.One of the benefits of this distribution is that the skewness can be introduced by a single parameter α.This parameter controls the shape of the distribution. For instance, when $\alpha = 0$, $f(x; \alpha)$ corresponds to the standard normal distribution. Plots of the univariate density (1) for $\alpha = -5, -2, 0, 2, 5$, given in Figure on the right, illustrate the effects of changing α on the shape of the density.



In general, a more flexible random variable can be built by incorporating location and scale parameters, ξ and ω, respectively such as $Y = \xi + \omega X$, where $X \sim \mathcal{SN}(\alpha)$, and this random variable Y is said to follow a skew normal distribution $Y \sim \mathcal{SN}(\xi, \omega, \alpha)$ the density of which is given as

$$f(\xi, \omega, \alpha) = \frac{2}{\omega}\phi\left(\frac{y-\xi}{\omega}\right)\Phi\left(\alpha\frac{y-\xi}{\omega}\right), \; y \in \mathbb{R}, \;\; \xi \in \mathbb{R}, \;\; \omega > 0, \;\; \alpha \in \mathbb{R} \tag{2}$$

An alternative representation of the skew-normal that is especially popular in modeling Bayesian analysis is its stochastic representation given by (Azzalini, 1986; Henze, 1986).The idea is if $Z \backsim \mathcal{TN}_{[0,\infty)}(0,1)$ and $\varepsilon \backsim \mathcal{N}(0,1)$ are independent, and $\delta \in (-1,1)$, then the stochastic representation for the skew normal random variable X is given by

$$X = \delta Z + \sqrt{1-\delta^2}\,\varepsilon \tag{3}$$

and that of $Y = \xi + \omega X$ is

$$Y = \xi + \omega X = \xi + \omega\left(\delta Z + \sqrt{1-\delta^2}\varepsilon\right) = \xi + \omega\delta Z + \omega\sqrt{1-\delta^2}\varepsilon \tag{4}$$

with $\delta = \frac{\alpha}{\sqrt{1+\alpha^2}}$

2.2 Hierarchical Bayesian Models

Considering the stochastic representation given above, the Hierarchical Bayesian structure for the univariate skew normal model is

$$\begin{aligned}
Y_i|Z_i,\xi,\omega,\alpha \backsim \mathcal{N}\left(\xi+\omega\delta Z_i, \omega^2(1-\delta^2)\right);\\
Z_i \sim \mathcal{N}_{[0,\infty)}\left(0,1\right),\ i=1,\ldots,n;\\
\xi \sim \mathcal{N}\left(\mu_\xi, \delta_\xi^2\right);\\
\omega^2 \sim Inverse-Gamma\left(a,b\right);\\
\alpha \sim \mathcal{N}\left(\mu_\alpha, \delta_\alpha^2\right) \qquad (Case-1);\\
\alpha \sim \mathcal{SN}\left(\xi_o,\omega_o,\alpha_o\right) \qquad (Case-2)
\end{aligned} \tag{5}$$

where $\mu_\xi, \mu_\alpha, \xi_o, \alpha_o \in \mathbb{R}$ and $a, b, \delta_\xi^2, \delta_\alpha^2, \omega_o > 0$ are hyperparameters of the model.even though the direct derivation of the marginal posterior of the parameters is complicated, the Gibbs Sampler can be used to generate the samples of the posterior.In order to implement the Gibbs sampler, we will use the data augmentation techniques.In order to benchmark the performance of the model, data will be simulated and examined. Posterior inference may also be highly dependent on the choice of the prior. We will also perform a sensitivity analysis on the parameters distribution.

References

[1] AZZALINI, A. (1985). "A class of distributions which includes the normal ones ." *Scandinavian Journal of Statistics*, **12**, 171-178.

[2] AZZALINI, A. (1986). "Further results on a class of distributions which includes the normal ones." *Statistica*, **46**, 199-208.

[3] HENZE, N. (1986). "A probabilistic representation of the skew-normal distribution." *Scandinavian Journal of Statistics*, **13**, 271-275.