Bayesian Filtering for Thermal Conductivity Estimation given Temperature Observations

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Abstract

International standards often require complex experimental layouts to estimate the thermal conductivity of materials, and they marginally take into account the uncertainty in the estimation procedure. In this paper, we propose a particle filtering approach coupled with a simple experimental layout for the real-time estimation of the thermal conductivity in homogeneous materials. Indeed, based on the heat equation, we define a state-space model for the temperature evaluation based on the unknown conductivity and apply a Rao-Blackwellized particle filter. Finally, the approach is validated considering heating and cooling cycles given to a specimen made up of PMMA in forced convection. Results show good estimates in accordance with the PMMA conductivity range, and computational times confirm the possibility of a real-time estimation.

Keywords: partial differential equations; heat equation; thermal conductivity estimation; particle filtering.

1 Introduction

International standards propose several methods for estimating the thermal conductivity of materials [8, 9, 10]. However, they require complex experimental layouts and marginally consider uncertainty in the estimation procedure. To improve the approach, a Bayesian estimation of conductivity coupled with a simple experimental layout has been recently proposed in [5]. This uses MCMC simulation for obtaining conductivity posterior density, and requires getting all temperature measurements before processing data.

In this paper, we propose a Rao-Blackwellized particle filter that allows real-time estimation instead of waiting for the entire dataset. This particle filter (PF) has been already applied to parameter estimation in ordinary differential equations [6, 7]; now we apply to the heat equation, i.e., a partial differential equation. The aim is to exploit the benefits of the simple experimental layout already proposed while integrating real-time estimation.

2 Method

2.1 State-space model

We consider the following nonlinear state-space model from spatial and temporal discretizations of the unidirectional heat equation, as in [5],

$$\mathbf{T}_j = \mathbf{T}_{j-1} + \mathbf{g}_j \lambda_0 + \mathbf{Q}_j \mathbf{\Delta w}_j, \tag{1}$$

$$\mathbf{o}_j = \mathbf{T}_j + \mathbf{H}_j \boldsymbol{\xi}_j, \tag{2}$$

where λ_0 is the thermal conductivity to estimate; $\mathbf{T}_j$ is the vector of true temperatures in each discretized point at time $j = 1, \ldots, F$; $\mathbf{o}_j$ is the corresponding vector of noisy temperature observations; $\mathbf{\Delta w}_j$ is a vector of independent Wiener processes; $\boldsymbol{\xi}_j$ is a vector of independent white noise processes. Moreover, $\mathbf{g}_j = \tau \mathbf{L}_{j-1}$, $\mathbf{Q}_j = \eta \mathbf{D}_{\mathbf{L}_{j-1}}$, and $\mathbf{H}_j = \varepsilon \mathbf{D}_{\mathbf{T}_j}$, where τ is the time interval of the temporal discretization, η and ε are the errors related to the noise processes, $\mathbf{D_X}$ denotes a diagonal matrix (with $d_{i,i} = x_i$), and $\mathbf{L}_{j-1}$ is defined as in [5].

Considering $\mathbf{\Delta T}_j = \mathbf{T}_j - \mathbf{T}_{j-1}$, Eq. (1) is rewritten as a linear-Gaussian state-space model with state variable λ_0 and observation vector $\mathbf{\Delta T}_j$,

$$\begin{aligned} \lambda_{0,j} &= \lambda_{0,j-1}, \\ \mathbf{\Delta T}_j &= \mathbf{g}_j \lambda_{0,j} + \mathbf{Q}_j \mathbf{\Delta w}_j. \end{aligned} \tag{3}$$

2.2 Rao-Blackwellized particle filter

The likelihood of $\lambda_{0,j}$ is Gaussian and we assume that λ_0 is *a priori* Gaussian; then, its posterior density at time j is also Gaussian with expected value $\hat{\lambda}_{0,j} = \int \lambda_0 \, p(\lambda_0 | \mathbf{\Delta T}_{1:j}) \, d\lambda_0$ and variance $P_j = \int (\lambda_0 - \hat{\lambda}_{0,j})^2 \, p(\lambda_0 | \mathbf{\Delta T}_{1:j}) \, d\lambda_0$. Due to the state-space model (3) linear in λ_0 and the Gaussian likelihood of $\lambda_{0,j}$, we can apply a Kalman filter [1, 4] to exactly compute the posterior distribution of λ_0.

In this paper, we apply a Rao-Blackwellized PF (RBPF) [2, 3] to jointly approximate the posterior distribution of the temperatures and estimate the unknown conductivity λ_0. The proposed PF handles S particles; hence, a set of S Kalman filters running in parallel is implemented. In this way,

$$p(\mathbf{T}_j | \mathbf{T}_{0:j-1}^{(i)}, \mathbf{o}_{1:j-1}) = \mathcal{N}(\mathbf{T}_j; \beta_j^{(i)}, \mathbf{B}_j^{(i)}), \tag{4}$$

where $\beta_j^{(i)} = \mathbf{g}_j^{(i)} \hat{\lambda}_{0,j-1}^{(i)} + \mathbf{T}_{j-1}^{(i)}$ and $\mathbf{B}_j^{(i)} = \mathbf{g}_j^{(i)} P_{j-1}^{(i)} \mathbf{g}_j^{(i)\top} + \tau \mathbf{Q}_j^{(i)} \mathbf{Q}_j^{(i)\top}$ (superscript $^{(i)}$ indicates the particle $\mathbf{T}_{0:j}^{(i)}$). Moreover, the posterior estimate of λ_0 at each time j is obtained using the statistics generated by the RBPF. In particular, the posterior mean and variance of λ_0 conditional on the observations $\mathbf{o}_{1:j}$ can be approximated as

$$\hat{\lambda}_{0,j}^S = \sum_{i=1}^{S} v_j^{(i)} \hat{\lambda}_{0,j}^{(i)} \qquad P_j^S = \sum_{i=1}^{S} v_j^{(i)} \left[\left(\hat{\lambda}_{0,j}^{(i)} - \hat{\lambda}_{0,j}^S \right)^2 + P_j^{(i)} \right]$$

where $v_j^{(i)}$ is the importance weight related to the particle $\mathbf{T}_j^{(i)}$.

In this way, a real-time estimation that evolves over time is achieved.

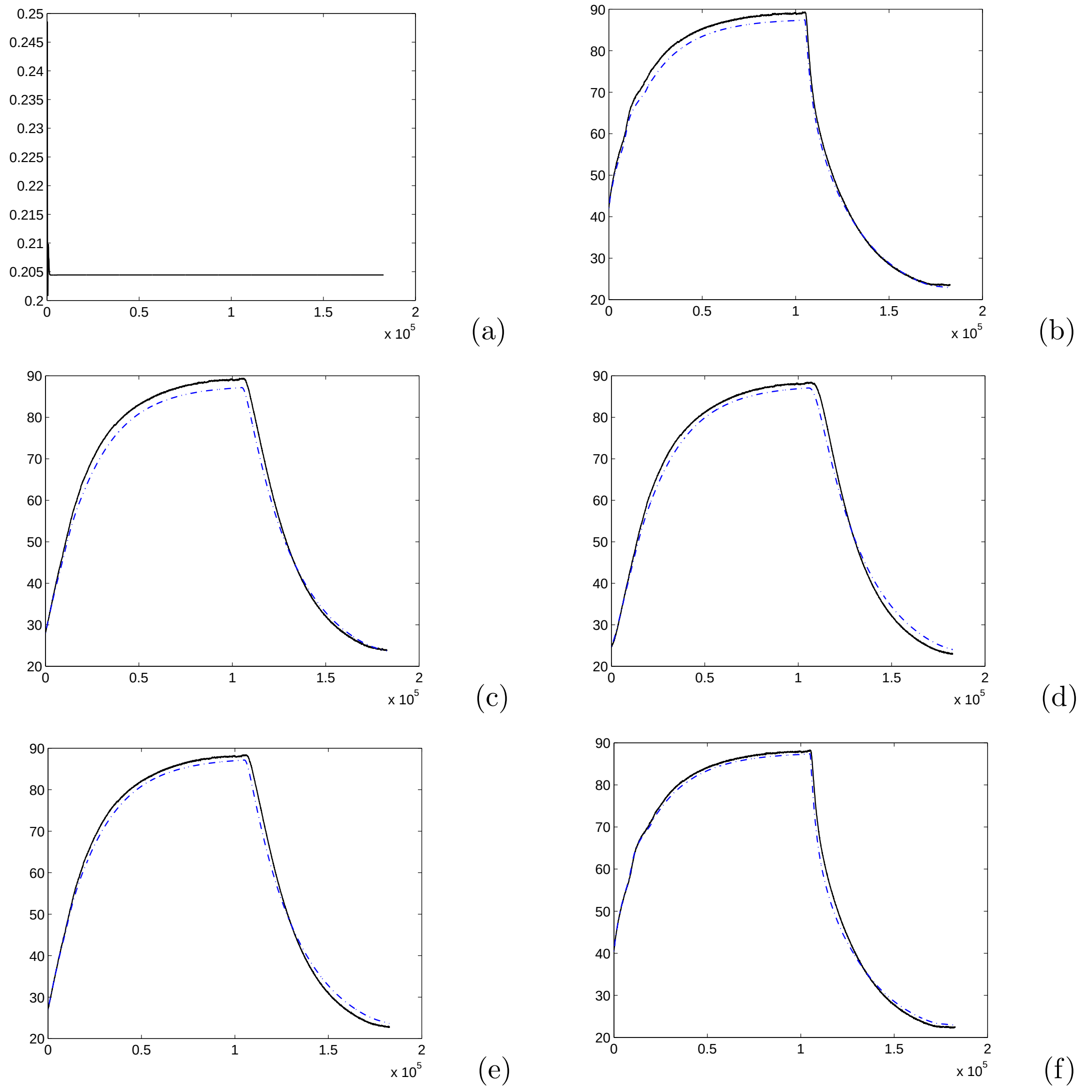

Figure 1: (a) mean $\lambda_{0,j}$ $[\frac{W}{mK}]$ over time instants j [sec]; (b)-(f) internal estimated (dotted lines) and acquired (solid lines) temperatures $[^oC]$ over time instants j [sec].

3 Application to the experimental layout and results

The proposed approach was applied to estimate the thermal conductivity of a polymer, for which a conductivity range of variability is known due to the polymer family, whereas the specific value within this range is unknown.

The experimental layout, as in [5], consists of a specimen made up of PMMA, with square faces (side of 20 cm) and thickness of 15 cm. Seven thermocouples are put in the specimen in the center of the square faces along a line: two external thermocouples on the square faces and five equally spaced internal thermocouples within the specimen. Finally, lateral rectangular faces are thermally insulated to guarantee unidirectional heat flow. Three experiments were conducted (A, B and C): each time a heating and cooling cycle, lasting about 40 hours, was given to the specimen in forced convection, within a range where the PMMA conductivity is constant (see [8], Part 5).

We applied the proposed RBPF with $S = 600000$ particles, $\tau = 60$ seconds, $\eta = 0.00015$ and $\varepsilon = 0.0015$.

We first validated the approach considering several simulated datasets of two hours: boundary and initial conditions are taken from the experiments, whereas internal temperatures are simulated with different conductivities within the PMMA range. Results show estimation errors always lower than 1%.

Then, we applied to the real datasets of the experiments, considering the entire experiment duration. Results in Tab. 1 show stable estimations among the experiments within the PMMA range and associated with low variances.

As for experiment A, we also show the plots of the λ_0 estimation over time (Fig. 1a) and the estimated internal temperatures compared to the acquired ones (Fig. 1b-f). Results show that the estimated value of λ_0 is stabilized after few instants, thus allowing low errors in temperature estimations already from the first points.

In conclusion, the proposed approach seems to be able to improve the estimation procedure of thermal conductivity in homogeneous materials by equipping the layout of [5] with a real-time estimation. Real-time estimation is confirmed, since the computational time to process a time step is lower than the time step, and the thermal conductivity value is stabilized after few time instants, allowing to accurately follow the temperature profile along the experiment.

	Experiment A	Experiment B	Experiment C
λ_0 mean $\left[\frac{W}{mK}\right]$	0.2044	0.2097	0.2169
λ_0 variance $\left[\frac{W^2}{m^2K^2}\right]$	$2.64 \cdot 10^{-14}$	$3.14 \cdot 10^{-14}$	$3.21 \cdot 10^{-14}$

Table 1: Posterior mean and variance of λ_0 estimates from the three experiments.

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